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Milnor fiber complexes for the exceptional Shephard groups

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Abstract

The symmetry group of a regular real polytope is a finite Coxeter group. The intersection of the unit sphere with the reflecting hyperplanes of the corresponding Coxeter arrangement induces a simplicial triangulation of the sphere, called the Coxeter complex. A Shephard group G is the symmetry group of a regular complex polytope. Orlik has shown that there can be associated to G a real simplicial complex which possesses many properties analogous to the Coxeter complex. Let f_1 be the G-invariant polynomial of minimal positive degree, and let $F = f_1^{-1}(1)$ be its Milnor fiber. Orlik showed that there is a complex Γ which is an equivariant strong deformation retract of F, is G-stable, is stratified by the associated reflection arrangement, and satisfies specific cell-counting formulas. His proof is existential; it does not give an explicit method of constructing the complex, though Orlik and Solomon did explicitly construct complexes for one infinite family of Shephard groups. Here we describe the construction of complexes for the remaining 15 "exceptional" Shephard groups. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let V be a vector space of dimension ℓ over the field $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $G \subset GL(V)$ be a finite irreducible reflection group. Let S be the $\mathbb{K}^*$ algebra of polynomial functions on V with the usual G-module structure $(gf)(v) = f(g^{-1}v)$. We choose a basis $z_1, \ldots, z_\ell$ (where $z_j = x_j + \mathrm{i}y_j$) for the dual space V^* so that $S = \mathbb{K}[z_1, \ldots, z_\ell]$. In the case when $\mathbb{K} = \mathbb{R}$, we use $x_1, \ldots, x_\ell$ as our basis for V^* to emphasize that fact.

Let $R = S^G$ be the subalgebra of S of G-invariant polynomials. Chevalley [1] showed that there exist polynomials $f_1, \ldots, f_\ell \in R$ such that $R = \mathbb{K}[f_1, \ldots, f_\ell]$. We call the set $\mathcal{B} = \{f_1, \ldots, f_\ell\}$ a set of *basic invariants* for G. Let $d_j = \deg f_j$. Although $\mathcal{B}$ is not

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uniquely determined by G, the set of degrees $\{d_1, \ldots, d_\ell\}$ is. We call this set the *basic degrees* of G and agree to order $\mathcal{B}$ so that

$$d_1 \leqslant \cdots \leqslant d_\ell.$$

The set $\{m_1, \ldots, m_\ell\}$ with $m_j = d_j - 1$ is the set of *exponents* of G.

If $r \in G$ is a reflection, then r fixes a hyperplane $H \subset V$. Let $\mathcal{A} = \mathcal{A}(G)$ be the set of reflecting hyperplanes (called the reflection *arrangement*), and let $L = L(\mathcal{A})$ be the set of intersections of elements of $\mathcal{A}$, partially ordered by reverse inclusion. Then L is a geometric lattice with minimal element V and rank function $r(Z) = \ell - \dim Z$ for $Z \in L$. Let μ be the Möbius function of L. For each $Z \in L$, define a polynomial $B_Z(t)$ by

$$B_Z(t) = (-1)^{\dim Z} \sum_{Y \geqslant Z} \mu(Z, Y)(-t)^{\dim Y}.$$

Let $F = f_1^{-1}(1)$ be the *Milnor fiber* of the first invariant of G. The topology of the Milnor fiber clearly depends on the choice of base field $\mathbb{K}$. If $\mathbb{K} = \mathbb{R}$ and G is a Coxeter group, then we may choose $f_1 = \sum x_j^2$ as the first invariant. In this case, the Milnor fiber is an $(\ell - 1)$-sphere in $\mathbb{R}^\ell$. The elements of $L(\mathcal{A})$ induce a simplicial triangulation of $S^{\ell-1}$ in the following way. The elements of $\mathcal{A}$ define a decomposition of V into a disjoint union of convex cones called the *facets* of $\mathcal{A}$. Two points $x, y \in V$ lie in the same facet if for every hyperplane $H \in \mathcal{A}$, either $x, y \in H$ or x and y lie on the same side of H. Each facet is a relatively open convex cone, and it is included in a unique minimal subspace of V called its *affine support*. The set of these supports is precisely the set $L(\mathcal{A})$. Each facet intersects the unit sphere $S^{\ell-1}$ in a relatively open spherical simplex. The union of these simplexes is a simplicial complex $\Gamma = \Gamma_G$, which is called the *Coxeter complex* of G. Orlik and Solomon investigated many of the properties and symmetries of the Coxeter complex in [8]. In particular, they showed that:

- Γ is G-stable.
- The action of G on Γ is transitive on top-dimensional simplexes and maps p-simplexes to p-simplexes.
- Γ is stratified by the intersection lattice: for any $Z \in L(\mathcal{A})$, the space $\Gamma_Z = \Gamma \cap Z$ is a subcomplex of Γ.
- The number of simplexes of a given dimension in Γ can be obtained from $L(\mathcal{A})$. In particular, the number of top-dimensional simplexes in Γ_Z is given by $B_Z(1)$ (see also [16]).

While Coxeter groups are most often thought of as real reflection groups, they may also be viewed as complex reflection groups by extending the base field $\mathbb{K}$ from $\mathbb{R}$ to $\mathbb{C}$ in the natural way. The collection of all finite irreducible complex reflection groups was classified by Shephard and Todd [15] and include these complexified versions of the Coxeter groups. Also included in this classification are the Shephard groups. A *Shephard group* is the symmetry group of a regular complex polytope. Since a Coxeter group may be defined as the symmetry group of a regular real polytope, the Shephard groups provide a complex analogue of the Coxeter groups. In the Shephard–Todd classification, there are two infinite families of (non-Coxeter) Shephard groups, together with fifteen so-called “exceptional” Shephard groups.

In [7], Orlik generalized the definition of the Coxeter complex to the Shephard groups. (Orlik and Terao have since generalized the notion further: see [14].) In this context, $\mathbb{K} = \mathbb{C}$, so the invariants are complex polynomials and the Milnor fiber F is a subset of $V = \mathbb{C}^\ell$. In [7], Orlik showed that for each Shephard group, the invariant f_1 of minimal positive degree is unique up to constant multiple. Moreover, f_1 has an isolated critical point at the origin. Then F is homotopy equivalent to a wedge of $(\ell - 1)$ spheres, and the number of spheres is given by the reduced Euler characteristic of F, or by the Milnor number of f_1: $\mu = m_1^\ell$ (see [5]). Orlik’s fundamental result was to show that for each Shephard group, there is a (real) cell complex with properties analogous to the Coxeter complex:

Theorem 1.1 (Orlik). *Let $G \subset GL(V)$ be a Shephard group, and f_1 a G-invariant polynomial of minimal positive degree. Let d_1 be the degree of f_1, and $m_1 = d_1 - 1$ the first exponent of G. Let $F = (f_1)^{-1}(1)$ be a Milnor fiber for f_1. Then there exists a simplicial complex $\Gamma \subset F$ which is an equivariant strong deformation retract of F with the following properties*:

(i) *For all $Z \in L$, the set $\Gamma_Z = \Gamma \cap Z$ is a subcomplex of Γ which is a strong deformation retract of $F \cap Z$.*

(ii) *Let Γ^p and Γ_Z^p denote the p-skeleton of each complex. Then $\Gamma_Z^p = \Gamma^p \cap Z$ for all p. Moreover, $\Gamma^p - \Gamma^{p-1} = \bigcup_{\dim Z = p+1} (\Gamma_Z^p - \Gamma_Z^{p-1})$ is a disjoint union.*

(iii) *Let $\gamma_k(Z)$ denote the number of k-simplexes in Γ_Z. If $\dim Z = p + 1$, then $\gamma_p(Z) = B_Z(m_1)$.*

We call the simplicial complex Γ a *Milnor fiber complex* for G. Although a Milnor fiber for an invariant of a Shephard group is a complex manifold in general, the complex Γ is a real simplicial complex, consisting of vertices, edges, and higher dimension cells. The strong deformation retraction allows for the passage between real and complex spaces.

Orlik’s proof in [7] was an existence proof; he did not explicitly construct the complexes Γ for the Shephard groups. In [8], Orlik and Solomon did construct the Milnor fiber complexes for one infinite family of Shephard groups, the monomial groups of the form $G(r, 1, \ell)$. The cyclic groups of order r make up the other infinite family of (non-Coxeter) Shephard groups. These are groups of complex rank $\ell = 1$, so the associated Milnor fiber complexes are sets of points and trivially satisfy the conditions of Theorem 1.1. It is the purpose of this paper to give explicit constructions of the complexes for the fifteen exceptional Shephard groups. If we define the *rank* of a Shephard group G as $\operatorname{rank}(G) = \dim_{\mathbb{C}}(V)$, then twelve of the exceptional groups are rank 2, two groups are rank 3, and there is a single group of rank 4. A complete enumeration of these groups (using the notation of Shephard and Todd) may be found in Table 1.

Let V/G be the orbit space and let $\gamma : V \to V/G$ be the canonical projection. Chevalley’s work shows that the map $\tau : V/G \to E_G = \mathbb{C}^\ell$ given by

$$\tau(Gv) = \bigl(f_1(v), \ldots, f_\ell(v)\bigr)$$

is a bijection. Then the composition $\pi = \tau\gamma : V \to E_G$ is the orbit map. Let $P = \pi(F)$ be the image of the Milnor fiber in the orbit space. If E_G has coordinates $T_1, \ldots, T_\ell$, then P is the affine subspace defined by $T_1 = 1$. In his proof of Theorem 1.1, Orlik showed that the image of Γ in the orbit space is topologically a real $(\ell - 1)$ simplex Λ, and is a strong deformation retract of P. Using spreads, he showed that Λ can be lifted inductively to $\mathbb{C}^\ell$ to obtain the complex Γ as a SDR of F.

The results of this paper may be summarized as follows. For each of the exceptional Shephard groups G, we describe $\Gamma = \Gamma_G$ and show that it is our desired complex by showing that its image under $\pi = \pi_G$ is $\Lambda = \Lambda_G$. (Here and throughout the paper, we use subscripts and occasionally superscripts when we wish to emphasize the group G.) In each case, Γ is a real model, which we obtain by defining certain real linear subspaces of V called *generating flats*. These flats together with the flats obtained from them under the group action of G intersect the Milnor fiber in the simplexes of Γ. The (real) p-cells of Γ are obtained by defining $(p+1)$-dimensional generating flats and their translates under the G-action. In this way, Γ is constructed inductively on cell dimension.

In Sections 2 and 3 we provide background which motivates the construction of the complexes. Section 4 provides a description of the constructions in the case of the exceptional rank 2 Shephard groups. A description of the general algorithm is given as well as necessary information for reproducing any of the 12 complexes of this type, and a detailed description of the construction for one of the rank 2 groups (G_4) is given as an example. Section 5 describes the constructions for the three higher dimension complexes. Again, a general recipe is provided, as well as a detailed example and necessary data for constructing the complexes for the other higher rank groups.

2. Stratification, discriminants and the discriminant matrix

Background on discriminants can be found in [6], [7, §2], and in [13]. Material in this section is drawn from these sources. Except where noted, G is a finite irreducible reflection group.

The intersection lattice $L = L(\mathcal{A})$ of $\mathcal{A}(G)$ provides a stratification of V: for $Z \in L$ define an arrangement in Z by

$$\mathcal{A}^Z = \{Z \cap H \mid H \in \mathcal{A} \text{ and } Z \not\subseteq H \text{ and } Z \cap H \neq \emptyset\}.$$

Let $N(Z)$ be the union of the hyperplanes in $\mathcal{A}^Z$, and let $M(Z) = Z \setminus N(Z)$ be the complement. Note that $N = N(V)$ is the union of the hyperplanes in $\mathcal{A}$, with $M = M(V)$ its complement. If Z is a dimension p element of $L(A)$, then $M(Z)$ is a relatively open p-submanifold in V. The collection $\{M(Z)\}_{Z \in L}$ is a set of pairwise disjoint such submanifolds whose union is V.

The orbit map π induces a stratification (the *orbit stratification*) on E_G using the images $\pi M(Z)$ of the stratification of V. The restriction $\pi|_M : M \to \pi M$ is a $|G|$-fold covering. The *branch locus* $\pi(N)$ is a hypersurface in E_G, and a defining polynomial for it may be obtained using the basic invariants. For each $H \in \mathcal{A}$, let e_H be the order of the cyclic subgroup fixing H, and let $\alpha_H \in V^*$ be a linear form with $H = \ker \alpha_H$. Then the form $\prod_{H \in \mathcal{A}} \alpha_H^{e_H}$ is an invariant form and can be expressed as a polynomial in the basic invariants. Thus we may define a polynomial $\Delta(T_1, \ldots, T_\ell; \mathcal{B})$ in the indeterminates $T_1, \ldots, T_\ell$ by

$$\Delta(f_1, \ldots, f_\ell; \mathcal{B}) = \prod_{H \in \mathcal{A}} \alpha_H^{e_H}.$$

This polynomial, called the *discriminant of G relative to $\mathcal{B}$* depends on the choice of invariant set $\mathcal{B}$.

The hypersurface

$$\pi(N) = \bigl\{(z_1, \ldots, z_\ell) \in E_G \mid \Delta(z_1, \ldots, z_\ell; \mathcal{B}) = 0\bigr\}$$

is called the *discriminant locus*. Just as $N = \bigcup_{Z > V} M(Z)$ is the union of all strata of dimension $\leqslant (\ell - 1)$ in V, so $\pi(N)$ is the union of all orbit strata of dimension $\leqslant (\ell - 1)$ in E_G.

Yet more can be said. Just as the branch locus is determined by the zero set of the discriminant, the lower dimensional strata in the orbit stratification are topologically determined by a refinement of the discriminant, called the discriminant matrix. Let $\mathsf{J}(\mathcal{B}) = \mathsf{J}(f_1, \ldots, f_\ell)$ be the Jacobian matrix defined by

$$\mathsf{J}(f_1, \ldots, f_\ell)_{i,j} = \frac{\partial f_j}{\partial x_i}.$$

Let $\Theta = \{\theta_1, \ldots, \theta_\ell\}$ be a set of basic derivations for G, i.e., a homogeneous basis for the R-module Der_S^G of G-invariant derivations. Let $\mathsf{M}(\Theta)$ be the coefficient matrix given by

$$\mathsf{M}(\theta_1, \ldots, \theta_\ell)_{i,j} = \theta_j(x_i).$$

Then $(\mathsf{J}^{\mathrm{T}}\mathsf{M})_{i,j} = \theta_j f_i$, so $(\mathsf{J}^{\mathrm{T}}\mathsf{M})$ is an $(\ell \times \ell)$ matrix with entries in R. The entries of this matrix may thus be expressed in terms of the basic invariants $f_1, \ldots, f_\ell$: there exist unique polynomials $\psi_{i,j} \in \mathbb{C}[T_1, \ldots, T_\ell]$ such that $(\mathsf{J}^{\mathrm{T}}\mathsf{M})_{i,j} = \psi_{i,j}(f_1, \ldots, f_\ell)$. Define the *discriminant matrix* $\mathsf{M}_\Delta(T_1, \ldots, T_\ell; \mathcal{B}, \Theta)$ by

$$\mathsf{M}_\Delta(T_1, \ldots, T_\ell; \mathcal{B}, \Theta)_{i,j} = \psi_{i,j}(T_1, \ldots, T_\ell).$$

In [6], Orlik showed that the determinant of the discriminant matrix provides an algebraic characterization of the orbit strata in E_G.

Theorem 2.1 [6, 3.8]. *For $0 \leqslant p \leqslant \ell - 1$, let $\mathcal{I}_p(\mathcal{B}, \Theta)$ be the ideal in $\mathbb{C}[T_1, \ldots, T_\ell]$ generated by the $(p + 1) \times (p + 1)$ minors of $\mathsf{M}_\Delta(T_1, \ldots, T_\ell; \mathcal{B}, \Theta)$. The variety of $\mathcal{I}_p(\mathcal{B}, \Theta)$ is the union of all strata of dimension $\leqslant p$ in the orbit stratification of E_G.*

In particular, $\det \mathsf{M}_\Delta(T_1, \ldots, T_\ell; \mathcal{B}, \Theta) \doteq \Delta(T_1, \ldots, T_\ell; \mathcal{B})$, so the discriminant matrix is a generalization of the discriminant.

In general, the discriminant matrix depends on the choice of Θ. However, if G is a Shephard group, then given a set of basic invariants $\mathcal{B} = \{f_1, \ldots, f_\ell\}$, the set of basic derivations $\Theta = \{\theta_1, \ldots, \theta_\ell\}$ can be chosen uniquely to satisfy the equation

$$\mathrm{Hess}(f_1)\theta_k = m_1 df_k \quad \text{for } 1 \leqslant k \leqslant \ell \tag{2.1}$$

(see [12]). In this way, the discriminant matrix for a Shephard group can be thought of as depending only upon $\mathcal{B}$. For the remainder of this paper we will assume that the basic derivations are chosen in this way.

In [2], Coxeter introduced a notation for the presentation of a Shephard group. He showed that to each such Shephard group G, there exist two sets of integers, $p_1, \ldots, p_\ell$ and $q_1, \ldots, q_{\ell-1}$ and distinguished (though not unique) generating reflections $r_1, \ldots, r_\ell$ such that G has a presentation with the following defining relations:

$$\begin{aligned} r_j^{p_j} &= 1, && \text{for } 1 \leqslant j \leqslant \ell, \\ r_j r_k &= r_k r_j, && \text{if } |j-k| \geqslant 2, \\ r_{j+1} r_j r_{j+1} \cdots &= r_j r_{j+1} r_j \cdots, && \text{if } 1 \leqslant j \leqslant \ell - 1, \end{aligned} \tag{2.2}$$

where there are q_j terms on each side of the last equation. To denote G, Coxeter used the symbol $p_1[q_1]p_2[q_2] \cdots p_{\ell-1}[q_{\ell-1}]p_\ell$. The Coxeter symbol is uniquely determined by G up to the replacement of $p_1[q_1]p_2 \cdots p_{\ell-1}[q_{\ell-1}]p_\ell$ with $p_\ell[q_{\ell-1}]p_{\ell-1} \cdots p_2[q_1]p_1$.

Since the Coxeter groups are generated by reflections of order 2, to each Shephard group we may associate a Coxeter group, by formally replacing the exponents p_j with the exponent 2 for all j:

Definition 2.2. Suppose $G \subset GL(V)$ is the Shephard group with symbol $p_1[q_1]p_2 \cdots p_{\ell-1}[q_{\ell-1}]p_\ell$. Let $W \subset GL(V)$ be the Coxeter group with symbol $2[q_1]2 \cdots 2[q_{\ell-1}]2$. We call W the *Coxeter group associated to G*.

A list of the Shephard groups G and their associated Coxeter groups W can be found in [13, Table 6.5, p. 267]. Note that W is also a Shephard group. When one compares the basic degrees of a Shephard group G and its associated Coxeter group W, the following universal property becomes apparent. For each pair (G, W), there is a rational number $\kappa = \kappa(G, W)$ such that

$$d_j^G = \kappa d_j^W \quad \text{for } 1 \leqslant j \leqslant \ell. \tag{2.3}$$

In [11], Orlik and Solomon showed that the discriminant matrices of G and W are closely related:

Theorem 2.3. *Let G be a Shephard group and let W be the associated Coxeter group. Given a set of basic invariants $\mathcal{B}_G$ there exists a set of basic invariants $\mathcal{B}_W$, and given a set of basic invariants $\mathcal{B}_W$ there exists a set of basic invariants $\mathcal{B}_G$ such that*

$$\mathsf{M}_\Delta^G(T_1, \ldots, T_\ell; \mathcal{B}_G) = \mathsf{M}_\Delta^W(\kappa T_1, \ldots, \kappa T_\ell; \mathcal{B}_W).$$

Theorem 2.3 together with Theorem 2.1 tells us that when G is a Shephard group, the orbit stratifications of G and W are thus identical.

3. Milnor fiber complexes for the Shephard groups

For the remainder of the paper, we restrict ourselves to the class of Shephard groups. Given such a group G, let $\ell = \operatorname{rank}(G)$, so that $V = \mathbb{C}^\ell$. Let W be the associated (complexified) Coxeter group, with κ the constant given by Eq. (2.3). Let $\mathcal{B}_G = \{f_1^G, \dots, f_\ell^G\}$ be a set of basic invariants for G, and choose a set of basic invariants $\mathcal{B}_W = \{f_1^W, \dots, f_\ell^W\}$ for W which satisfy Theorem 2.3. Since W is a Shephard group, f_1^W is unique up to constant multiple, and without loss of generality, we may choose $f_1^W = \sum z_k^2$.

Essentially, Orlik's proof of Theorem 1.1 proceeds as follows. Let $F_W = (f_1^W)^{-1}(\kappa)$ be a (complex) Milnor fiber. Then F_W is homotopy equivalent to an ℓ-sphere with two points removed, and there is an equivariant strong deformation retraction from F_W onto the Coxeter complex Γ_W (which is an $(\ell-1)$-sphere). This induces a strong deformation retraction in the orbit space E_W from $P_W = \pi_W(F_W)$ onto $\Lambda_W = \pi_W(\Gamma_W)$. By the choice of basic invariants for G and W, there is a bijection $\rho : E_W \to E_G$ of stratified spaces given by $\rho(x) = \frac{x}{\kappa}$. Let $P_G = \rho(P_W)$ and $\Lambda_G = \rho(\Lambda_W)$. Then ρ induces a strong deformation retract from P_G onto Λ_G. Finally, the Milnor fiber complex for G is defined inductively: $\Gamma_G \subset V$ is defined by lifting $\Lambda_G \subset E_G$, and the strong deformation retraction from P_G onto Λ_G is lifted to obtain one from F_G onto Γ_G.

There are several useful consequences of the proof of Theorem 1.1. Since W acts transitively on the top-dimensional simplexes of Γ_W, Λ_W is topologically an $(\ell-1)$-dimensional simplex in E_W. Since P_W is the affine plane $T_1 = \kappa$ in E_W, the boundary of this simplex is given by $\Delta_W(\kappa, T_2, \dots, T_\ell; \mathcal{B}_W) = 0$. Since the stratified spaces E_G and E_W are identical (up to scaling), Orlik's theorem, together with Theorem 2.3 give us the following results:

Corollary 3.1.

(1) *Λ_G is topologically an $(\ell-1)$-dimensional simplex in E_G, where $\ell = \dim V$ is the rank of G. Thus the action of G on Γ_G is transitive on top-dimensional simplexes.*

(2) *For $0 \leqslant p \leqslant \ell - 1$, the union of the $(p-1)$-dimensional cells of Λ_G is given by the variety of the ideal generated by the $(p+1) \times (p+1)$ minors of $\mathsf{M}_\Delta^G(1, T_2, \dots, T_\ell; \mathcal{B}_G)$. In particular, the boundary of Λ_G is defined by the equation $\Delta_G(1, T_2, \dots, T_\ell; \mathcal{B}_W) = 0$.*

(3) *Since W is a Coxeter group, it has a real representation so we may choose invariants $\mathcal{B}_W$ of W so that $\Delta_W(\kappa, T_2, \dots, T_\ell; \mathcal{B}_W)$ is a polynomial with real coefficients. If we choose the invariants $\mathcal{B}_G$ of G correspondingly, then $\Delta_G(1, T_2, \dots, T_\ell; \mathcal{B}_G)$ is also a polynomial with real coefficients. In this case, Λ_W lies entirely in the real part of the coordinates of E_W, and Λ_G lies entirely within the real part of the coordinates of E_G.*

Corollary 3.1 allows us to explicitly describe our Milnor fiber complexes. In general, for each exceptional Shephard group G, we shall choose invariants so that Λ_G is a real

ℓ-simplex. Once Γ_G has been defined, we verify that its image under the orbit map π_G is Λ_G, which shows that Γ_G is the required Milnor fiber complex.

From Theorem 1.1, the number of simplexes of a given dimension in Γ_G is found by considering the polynomial $B_Z(t)$. For ease of notation, let $v_k = \gamma_k(V)$ be the total number of simplexes of dimension k in the full complex Γ. Then part (iii) of Theorem 1.1 tells us that v_k is given by

$$v_k = \sum_{\substack{Z\in L,\\ \dim Z=k+1}} \gamma_k(Z) = \sum_{\substack{Z\in L,\\ \dim Z=k+1}} B_Z(m_1), \tag{3.1}$$

where m_1 is the minimal exponent of G.

In [9,10], Orlik and Solomon computed the polynomials $B_Z(t)$ for all irreducible reflection groups G and for all $Z \in L(\mathcal{A})$. This allows us to calculate $\gamma_k(Z)$ and hence the value of v_k in all cases.

For the rank 2 Shephard groups, $B_H(t) = 1 + t$ for all $H \in \mathcal{A}$, so the total number of vertices in the complex is $|\mathcal{A}| \cdot (1 + m_1) = |\mathcal{A}| \cdot d_1$. Since the action of the group will be transitive on the 1-cells, the total number of 1-cells is $|G|$.

For the rank 3 and 4 groups, we refer to [13, Appendix C] for data on the lattices of the corresponding arrangements. This data allows us to compute v_k in all cases. For example,

Table 1
Cell counts for the Milnor fiber complexes for the exceptional Shephard groups

Group	v_0	v_1	v_2	v_3	Milnor number
G_4	16	24			9
G_5	48	72			25
G_6	40	48			9
G_8	48	96			49
G_9	144	192			49
G_{10}	168	288			121
G_{14}	120	144			25
G_{16}	240	600			361
G_{17}	840	1200			361
G_{18}	960	1800			841
G_{20}	240	360			121
G_{21}	600	720			121
G_{25}	126	648	648		125
G_{26}	342	1512	1296		125
G_{32}	4800	71280	207360	155520	14641

we note from [13, Table C.13] that $L(\mathcal{A}(G_{32}))$ has two orbits of rank 2 elements, and for each $Z \in L(\mathcal{A}(G_{32}))$ with $\operatorname{rank}(Z) = 2$, $B_Z(t) = (t+1)(t+7)$. Thus, since there are 330 elements of rank 2 in $L(\mathcal{A}(G_{32}))$, we have $v_1 = 330 \cdot (m_1+1) \cdot (m_1+7) = 71280$. Using calculation methods such as these, we obtain the cell counts found in Table 1.

4. The rank 2 exceptional Shephard groups

If G is a rank 2 group, then Γ is a 1-complex, and its image in $E_G = \mathbb{C}[T_1, T_2]$ is an arc joining two vertices. We choose a specific invariant set $\mathcal{B} = \{f_1, f_2\}$ for G so that $\pi(\Gamma)$ is the interval $\Lambda = [-1, 1]$ on the real axis in the T_2 coordinate (and in the subspace $T_1 = 1$). We then define Γ and show $\pi(\Gamma) = \Lambda$.

By Theorem 1.1, the vertices of Γ are found by intersecting the hyperplanes of $\mathcal{A}$ with F. The vertices come in two orbits, mapping to $P = (1, -1)$ and $Q = (1, 1)$, respectively, in the orbit space.

To describe the edges of the complex, we first identify $V = \mathbb{C}^2$ with $\mathbb{R}^4$ (using the coordinates $z_j = x_j + \mathrm{i}y_j$ for $j = 1, 2$). We then define specific real 2-dimensional subspaces of $\mathbb{C}^2 \approx \mathbb{R}^4$ which intersect F in curves which compose the edges of Γ:

Definition 4.1. Let $X \subset \mathbb{R}^4$ be a two-dimensional linear subspace. X is a *Γ-generating 2-flat associated to* $\{f_1, f_2\}$ provided that:

(i) $F \cap X \neq \emptyset$.
(ii) $\operatorname{Im} f_1|_X = 0$.
(iii) At least one branch of the algebraic curve $F \cap X$ contains points from both orbits of the vertices of Γ.
(iv) $\operatorname{Im} f_2|_X = 0$.

Conditions (i) and (ii) imply that $F \cap X$ is an algebraic curve in $\mathbb{R}^4$. Since F is a defined by two real restrictions ($\operatorname{Re} f_1 = 1$ and $\operatorname{Im} f_1 = 0$) in $\mathbb{R}^4$, one would expect that a generic two (real) dimensional linear space X would intersect F in a 0-dimensional set (i.e., a collection of points.) However, the condition that $\operatorname{Im} f_1|_X = 0$ ensures that X does not intersect F transversally, so $F \cap X$ is a (real) 1-dimensional set (i.e., a real algebraic curve). Conditions (iii) and (iv) of the definition ensure that the compact pieces of that curve are mapped onto Λ in the orbit space. Our fundamental result is that for each rank 2 exceptional Shephard group, such a 2-flat exists:

Theorem 4.2. *For each rank* 2 *Shephard group, there exist invariants* $\{f_1, f_2\}$ *and a set* $X \subset \mathbb{R}^4 \approx \mathbb{C}^2$ *such that* X *is a* Γ*-generating* 2*-flat associated to* $\{f_1, f_2\}$.

Table 2 gives our desired invariants $\{f_1, f_2\}$ for the rank 2 Shephard groups as well as the generating 2-flats. We use Shephard and Todd's [15] notation and representations of the rank 2 Shephard groups. The invariants are defined in terms of Klein's polynomials (see [3]).

Table 2
Invariants for the rank 2 Shephard groups and the corresponding Γ-generating 2-flats

Group	f_1	f_2	Γ-generating 2-flat
G_4	Φ	$\sqrt{6\sqrt{3}}(1+\mathrm{i})t$	$x_1=0,\ x_2=y_2$
G_5	$(1+\mathrm{i})t$	$\frac{1}{9}\sqrt{3}\Phi^3-(1+\mathrm{i})^2t^2$	$x_1=0,\ x_2=y_2$
G_6	Φ	$24\mathrm{i}\sqrt{3}t^2-\Phi^3$	$x_1=0,\ x_2=y_2$
G_8	W	χ	$y_1=0,\ y_2=0$
G_9	W	$2\chi^2-W^3$	$y_1=0,\ y_2=0$
G_{10}	χ	$2W^3-\chi^2$	$x_2=0,\ x_1=y_1$
G_{14}	$(1+\mathrm{i})t$	$\frac{2}{27}\chi^2-(1+\mathrm{i})^4t^4$	$x_1=0,\ x_2=y_2$
G_{16}	$-H$	T	$y_1=0,\ y_2=0$
G_{17}	$-H$	$2T^2+H^3$	$y_1=0,\ y_2=0$
G_{18}	T	$2H^3+T^2$	$y_1=0,\ y_2=0$
G_{20}	f	$\frac{1}{96}\sqrt{3}T$	$y_1=0,\ y_2=0$
G_{21}	f	$\frac{1}{864}T^2-f^5$	$y_1=0,\ y_2=0$

Once a Γ-generating 2-flat X has been obtained, one can consider the action of G on X. Since f_1 and f_2 are G-invariant polynomials, the Milnor fiber F is G-stable. Thus, given $g \in G$ and $x \in F \cap X$, the point gx lies in both F and gX. Thus condition (i) of Definition 4.1 is satisfied by gX. The invariance of f_1 and f_2 also ensures that gX satisfies conditions (ii) and (iv). If $F \cap X$ contains an edge $\overline{P_1Q_1}$ connecting vertices P_1 and Q_1 from two different orbits in Γ, then $\pi(\overline{P_1Q_1}) = \Lambda = \pi(g\overline{P_1Q_1})$. The flat gX therefore contains vertices gP_1 and gP_2 connecting an edge of Γ and condition (iv) of Definition 4.1 is satisfied. Thus, we have the following:

Corollary 4.3. *If X is a Γ-generating 2-flat associated to $\{f_1, f_2\}$, then for each $g \in G$, gX is a Γ-generating 2-flat associated to $\{f_1, f_2\}$.*

Given one Γ-generating 2-flat X, all of the edges of Γ can be found by considering $\{F \cap gX \mid g \in G\}$. In this way, X and its translates under the action of G generate Γ.

4.1. Example: G_4

The group $G = G_4$ consists of 24 elements. The arrangement $\mathcal{A} = \mathcal{A}(G)$ consists of 4 complex hyperplanes, all in the same orbit. A defining polynomial for $\mathcal{A}$ is given by

$$Q(\mathcal{A}) = \Psi(z_1, z_2) = z_1^4 - 2\mathrm{i}\sqrt{3}\, z_1^2 z_2^2 + z_2^4.$$

We choose as our basic invariants for G the set $\mathcal{B} = \{f_1, f_2\}$, where

$$f_1(z_1, z_2) = \Phi(z_1, z_2) = z_1^4 + 2\mathrm{i}\sqrt{3}\, z_1^2 z_2^2 + z_2^4, \quad \text{and}$$
$$f_2(z_1, z_2) = \sqrt{6\sqrt{3}}(1+\mathrm{i})t(z_1, z_2) = \sqrt{6\sqrt{3}}(1+\mathrm{i})z_1 z_2\big(z_1^4 - z_2^4\big).$$

Thus, the basic degrees of G are $d_1 = 4$, $d_2 = 6$, and the corresponding exponents of G are $m_1 = 3$, $m_2 = 5$. The Milnor fiber $F = f_1^{-1}(1)$ is topologically an oriented surface of genus 3 with 4 points removed and the Milnor number of F is $m_1^2 = 9$.

With the choice of $\mathcal{B}$, by direct calculation one can find the discriminant matrix M_Δ satisfying Eq. (2.1) as well as the corresponding discriminant polynomial:

$$\mathsf{M}_\Delta(T_1, T_2; \mathcal{B}) = \begin{pmatrix} 4T_1 & 6T_2 \\ 6T_2 & 9T_1^2 \end{pmatrix}, \quad \text{so}$$
$$\Delta(T_1, T_2; \mathcal{B}) \doteq \det \mathsf{M}_\Delta(T_1, T_2; \mathcal{B}) = 36T_1^3 - 36T_2^2. \tag{4.1}$$

The zero set of $\Delta(1, T_2; \mathcal{B})$ is $T_2 = \pm 1$. Theorem 2.1 then tells us that Λ_G is a 1-simplex whose endpoints are the points $P = (1, -1)$ and $Q = (1, 1)$ in $E_G = \mathbb{C}^2$.

Since $d_1 = 4$ and $|\mathcal{A}| = 4$, there are $4 \cdot 4 = 16$ vertices in Γ_G, obtained by intersecting the hyperplanes of $\mathcal{A}$ with F. There are two orbits of these vertices, which are sent, respectively, to P and Q in E_G. Eight of the vertices (call them $P_1, \ldots, P_8$) are mapped to P under π_G, and eight vertices ($Q_1, \ldots, Q_8$) are mapped to Q. For simplicity, let constants c_1 and c_2 be given by

$$c_1 = \frac{1}{2\sqrt{2}}\left(1 - \frac{1}{\sqrt{3}}\right)\big(9 + 6\sqrt{3}\big)^{1/4} \approx 0.3136, \quad \text{and}$$
$$c_2 = \frac{1}{\sqrt{6}}\big(9 + 6\sqrt{3}\big)^{1/4} \approx 0.8567.$$

Then, using $\mathbb{C}^2$ coordinates, the 16 vertices of the complex are given by

$$\begin{aligned}
&P_1 = (-\mathrm{i}c_2, -c_1 - \mathrm{i}c_1), && P_2 = (\mathrm{i}c_2, c_1 + \mathrm{i}c_1),\\
&P_3 = (c_1 + \mathrm{i}c_1, -\mathrm{i}c_2), && P_4 = (-c_1 - \mathrm{i}c_1, \mathrm{i}c_2),\\
&P_5 = (c_1 - \mathrm{i}c_1, c_2), && P_6 = (-c_1 + \mathrm{i}c_1, -c_2),\\
&P_7 = (-c_2, c_1 - \mathrm{i}c_1), && P_8 = (c_2, -c_1 + \mathrm{i}c_1),\\
&Q_1 = (-\mathrm{i}c_2, c_1 + \mathrm{i}c_1), && Q_2 = (\mathrm{i}c_2, -c_1 - \mathrm{i}c_1),\\
&Q_3 = (c_1 - \mathrm{i}c_1, -c_2), && Q_4 = (-c_1 + \mathrm{i}c_1, c_2),\\
&Q_5 = (-c_2, -c_1 + \mathrm{i}c_1), && Q_6 = (c_2, c_1 - \mathrm{i}c_1),\\
&Q_7 = (c_1 + \mathrm{i}c_1, \mathrm{i}c_2), && Q_8 = (-c_1 - \mathrm{i}c_1, -\mathrm{i}c_2).
\end{aligned}$$

To describe the edges of the complex for G_4, we define a Γ-generating 2-flat. Let $X = X_1^{(2)} = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \mid x_1 = 0, x_2 = y_2\}$. (We use the exponent on X to emphasize the real dimension). Since $f_1(\mathrm{i}y_1, y_2 + \mathrm{i}y_2) = y_1^4 + 4\sqrt{3}y_1^2 y_2^2 - 4y_2^4$, we have that $F \cap X$ is the curve $y_1^4 + 4\sqrt{3}y_1^2 y_2^2 - 4y_2^4 = 1$ in X (projecting into the y_1 and y_2

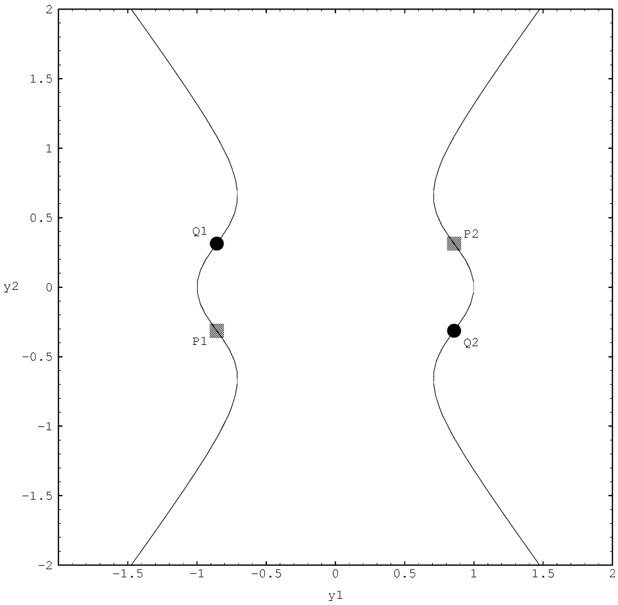

Fig. 1. The curve $F \cap X$ for the group G_4 (projected into (y_1, y_2) coordinates).

coordinates). There are four vertices (P_1, P_2, Q_1, and Q_2) of the complex which lie in X, and hence on the curve. Fig. 1 shows the curve $F \cap X$ as well as these vertices.

Restricting our second invariant to X gives us the polynomial

$$f_2(\mathrm{i}y_1, y_2 + \mathrm{i}y_2) = -2\sqrt{6\sqrt{3}}\big(y_1^5 y_2 + 4y_1 y_2^5\big) \in \mathbb{R}[y_1, y_2].$$

Since $f_2|_X$ is a polynomial with real coefficients, we have that each branch of the curve $F \cap X$ shown in Fig. 1 maps onto the real axis of $\mathbb{C}$ in the T_2 coordinate of E_G. That is, if $v = (x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2) \in F \cap X$, then $\pi_G(v) = (1, -2\sqrt{6\sqrt{3}}(y_1^5 y_2 + 4y_1 y_2^5))$. The arc $\overline{P_1 Q_1}$ in $F \cap X$ maps to the edge $\overline{PQ}$ in the T_2 coordinate in the orbit space, since $\pi_G(P_1) = P$ and $\pi_G(Q_1) = Q$ and the restriction $\pi_G|_X$ is real. Similarly, the arc $\overline{P_2 Q_2}$ in $F \cap X$ maps to the edge $\overline{PQ}$ in the orbit space.

There are 24 edges in the Milnor fiber complex for G_4. $\overline{P_1 Q_1}$ and $\overline{P_2 Q_2}$ are two of them. To find the other 22 arcs, we consider the orbit of the arc $P_1 Q_1$ under the action of G. Alternatively, we consider the orbit of the generating 2-flat X, intersecting its images under the action of G_4 with F to get the other arcs in the complex. Since X contains two arcs in the complex, and there are 24 arcs total, there are twelve 2-flats in the orbit of X. Call these 2-flats $X_1^{(2)}, \ldots, X_{12}^{(2)}$, with $X_1 = X$. For each 2-flat X_j $(1 \leqslant j \leqslant 12)$, we have that $\operatorname{Im} f_1|_{X_j} = 0$. Each X_j intersects F in a curve similar to that of Fig. 1. Each branch of the curve contains a vertex of each orbit type and hence an arc of the complex. Thus each flat X_j contains two edges of the complex. There are four vertices in each flat, so each vertex is contained in three of the 2-flats, and thus in three different edges of the complex.

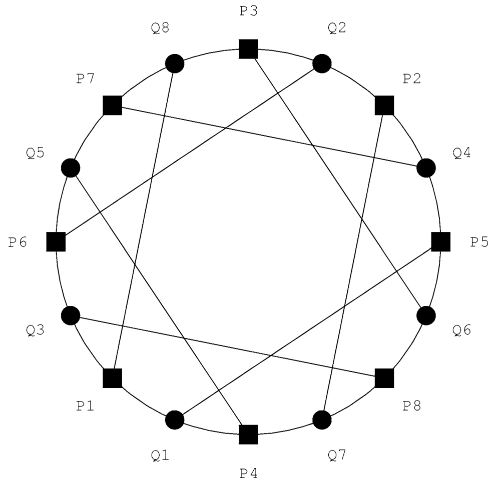

Fig. 2. An abstract depiction of the Milnor fiber complex Γ_{G_4}. Note that the only intersection points of the edges are at the 16 vertices.

An abstract description of the complex Γ_G can be found in [7, p. 153]. In this description, the 16 vertices of the complex are arranged on a circle, with each vertex attached to one other non-adjacent vertex as well. Fig. 2 is the representation of this form of the complex in our notation, showing all 16 vertices and 24 edges of the complex. (In the figure, the only intersection points of the edges are at the 16 vertices.)

This completes our description of the complex for G_4.

5. The rank 3 and 4 exceptional Shephard groups

There are two exceptional Shephard groups of rank 3, denoted G_{25} and G_{26} in the notation of Shephard and Todd, and there is a single exceptional group of rank 4, denoted G_{32}. In this section we describe a method for constructing the Milnor fiber complexes for these groups. The method is a direct generalization of the techniques used to construct the complexes in the rank 2 case. In the higher dimensions, just as in the theoretical construction given in Theorem 1.1, the construction is done inductively on $\dim Z$, where $Z \in L(\mathcal{A})$ is an element in the lattice of the corresponding arrangement. One notes from Table 1 that the number of cells of each dimension for these complexes is large. Giving a point-by-point description is not practical, and even giving a reasonably complete description of the three complexes is somewhat cumbersome. Instead we describe our method, then give a careful description of the complex for G_{26} as a means of illustrating the method. We then provide the necessary information for G_{25} and G_{32} so that the interested reader can readily produce the complex at the level of detail desired.

Let G be one of the three groups G_{25}, G_{26}, or G_{32} with $\ell = \operatorname{rank} G = \dim V$ (so that $\ell = 3$ or 4). Then $\Lambda = \Lambda_G$ is topologically an $(\ell - 1)$-simplex in $E_G = \mathbb{C}^\ell \approx \mathbb{R}^{2\ell}$. By Corollary 3.1, we may choose invariants $\mathcal{B} = \{f_1, \dots, f_\ell\}$ so that Λ lies in the real subspace

$\mathbb{R}^\ell \subset \mathbb{C}^\ell = E_G$. The vertices of Γ are obtained by intersecting the complex lines (i.e., the dimension 1 elements) of $L(\mathcal{A})$ with F. For a rank ℓ group, there will be ℓ orbits of vertices, corresponding to the ℓ 0-simplexes of Λ.

In general, for each complex dimension $p+1$ ($0 \leqslant p \leqslant \ell - 1$) element $Z \in L(\mathcal{A})$, we construct the real p-subcomplex $\Gamma_Z = \Gamma \cap Z$. The union

$$\bigcup_{\substack{Z \in L(\mathcal{A}) \\ \dim Z = p+1}} \Gamma_Z$$

is the p-skeleton of Γ. Alternatively, if there are k different orbits of $(p+1)$-dimensional lattice elements, one can choose representative elements $Z_1, \dots, Z_k$ for each orbit type and compute Γ_{Z_j} for each j. Then

$$\bigcup_{\substack{1 \leqslant j \leqslant k \\ g \in G}} g(\Gamma_{Z_j})$$

is the p-skeleton of Γ. It is this approach which we use.

If $p = 1$, then we choose representatives $Z_1, \dots, Z_k$ for the respective orbits of (complex) dimension 2 elements of $L(\mathcal{A})$, and construct each subcomplex Γ_{Z_j} as in the rank 2 case above. In general $\pi(\Gamma_{Z_j})$ is not a straight line as in the rank 2 case, but lies in the subspace $\{(1, T_2, \dots, T_\ell) \mid T_j \in \mathbb{R}\} \subset E_G$. Since there are multiple orbits of edges of Γ, constructing the 1-skeleton of Γ can require finding more than one orbit of 2-flats.

For $p > 1$, we require an extension of Definition 4.1.

Definition 5.1. Let Z be a complex $(p+1)$-dimensional element of $L(\mathcal{A})$. A real $(p+1)$-dimensional linear subspace X of Z is a *Γ_Z-semigenerating $(p+1)$-flat associated to $\mathcal{B}$* provided that:

(i) $F \cap X \neq \emptyset$.

(ii) $\operatorname{Im} f_j|_X = 0$ for all j with $1 \leqslant j \leqslant \ell$.

(iii) At least one component of the real p-dimensional algebraic hypersurface $F \cap X$ contains $(p-1)$-dimensional cells of Γ_Z which span a p-simplex.

As in Definition 4.1, the requirement that $\operatorname{Im} f_1|_X = 0$ ensures that $F \cap X$ is indeed a p-dimensional hypersurface. Since F is a real codimension 2 surface, a transverse intersection of F with a codimension $2\ell - (p+1)$ surface will be a codimension $2\ell - (p-1)$ surface, i.e., a real $(p-1)$-dimensional space. However, since X lies in the zero set of $\operatorname{Im} f_1$, $F \cap X$ is not a transverse intersection and $F \cap X$ is indeed a p-dimensional hypersurface. The condition that $\operatorname{Im} f_j|_X = 0$ for $j > 1$ implies that $\pi(F \cap X)$ lies in the real part of E_G. If $F \cap X$ also contains $(p-1)$-dimensional cells of Γ_Z spanning a p-simplex, then $F \cap X$ in fact contains a p-simplex of Γ_Z. We use the term "semigenerating" because in general more than one such flat is required to generate Γ_Z.

Just as in the rank 2 case, for the rank 3 and 4 groups, we can find flats which generate our complex:

Theorem 5.2. *For each rank* 3 *or* 4 *exceptional Shephard group* G *and for each* $Z \in L(\mathcal{A}(G))$ *with* $\dim Z = (p+1)$, *there exist real* $(p+1)$-*dimensional linear subspaces* $X_1, \ldots, X_k$ (*for some* k) *such that*:

(i) *for each* j, X_j *is a* Γ_Z-*semigenerating* $(p+1)$-*flat for* Γ_Z.

(ii) *For each* p-*simplex* σ *of* Γ_Z *there is a* j *such that* σ *is contained in* X_j.

For each of the higher rank groups, we choose the flat invariants of [13, Appendix B] as our set of basic invariants. This is particularly convenient since the discriminant matrices relative to these invariants have been computed already. For G_{25} and G_{26}, we use group representations given in [15], while for G_{32}, we use the representation of [15] but with the coordinate interchange $z_1 \leftrightarrow z_4$, $z_2 \leftrightarrow z_3$.

For each $Z \in L(\mathcal{A})$, we find a set of Γ_Z-semigenerating $(p+1)$-flats. One can then check that the subcomplex Γ_Z maps to the appropriate cells of Λ. Table 3 gives the Γ_Z-semigenerating $(p+1)$-flats for G_{25}, G_{26}, and G_{32} and for $1 \leqslant p \leqslant (\ell-1)$. The

Table 3
Milnor fiber complex semigenerating $(p+1)$-flats for the rank 3 and 4 Shephard groups

G	p	$Z \in L(\mathcal{A})$	Semigenerating $(p+1)$-flat(s) in Z	Number of orbits of p-simplexes
G_{25}	1	$z_3 = 0$	$y_1 = y_2 = 0$	2
			$x_1 + x_2 = y_1 - y_2 = 0$	1
	2	$\mathbb{C}^3$	$y_1 = y_2 = y_3 = 0$	1
G_{26}	1	$z_3 = 0$	$y_1 = y_2 = 0$	1
			$x_1 + x_2 = y_1 - y_2 = 0$	1
	1	$z_2 - z_3 = 0$	$y_1 = y_2 = 0$	1
	2	$\mathbb{C}^3$	$y_1 = y_2 = y_3 = 0$	1
G_{32}	1	$z_3 = 0, z_4 = 0$	$y_1 = y_2 = 0$	1
			$x_1 = x_2 = 0$	1
			$\begin{cases} -\frac{1}{2}x_1 + \frac{1}{2}\sqrt{3}y_1 - x_2 = 0, \\ \frac{1}{2}\sqrt{3}x_1 + \frac{1}{2}y_1 - y_2 = 0 \end{cases}$	1
	1	$\begin{cases} z_2 + z_3 = 0, \\ z_4 = 0 \end{cases}$	$y_1 = y_2 = 0$	3
	2	$z_4 = 0$	$y_1 = y_2 = y_3 = 0$	2
			$x_1 = x_2 = x_3 = 0$	2
	3	$\mathbb{C}^4$	$y_1 = y_2 = y_3 = y_4 = 0$	1

table also indicates the number of orbits of p-simplexes contained in each $(p+1)$-flat. For a rank ℓ group, the total number of orbits of p-cells is $(\frac{\ell}{p+1})$, since this is the number of p-cells in the simplex Λ.

We now give a detailed description of the complex for G_{26}, followed by sketches of the constructions for both G_{25} and G_{32}.

5.1. Example: G_{26}

The group $G = G_{26}$ is a rank 3 Shephard group consisting of 1296 elements. We use Shephard and Todd's representations for this group. As a subgroup of $GL(\mathbb{C}^3)$, G_{26} can be represented as the group generated by three reflections: the reflections r_1 and r_2 found in [15, Eq. 10.3], together with the reflection s given in [15, 10.4].

The groups G_{25} and G_{26} are closely related, arising as symmetry groups of the Hessian configuration (see [13, Example 6.30]). Using the reflections given above, G_{25} can be viewed as an order 2 subgroup of G_{26}, generated by reflections r_1, r_2, and r_3, where $r_3 = sr_1s$.

The reflection arrangement $\mathcal{A} = \mathcal{A}(G)$ consists of 21 hyperplanes. A defining polynomial for $\mathcal{A}$ is

$$Q(\mathcal{A}) = z_1z_2z_3 \prod_{0\leqslant j,k\leqslant 2} \left(z_1 + \omega^j z_2 + \omega^k z_3\right) \prod_{1\leqslant j<k\leqslant 3} \left(z_j^3 - z_k^3\right),$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$.

In our representation of G_{25}, a defining polynomial for $\mathcal{A}(G_{25})$ is given by the relation $Q(\mathcal{A}(G_{26})) = Q(\mathcal{A}(G_{25})) \cdot \prod_{1\leqslant j<k\leqslant 3}(z_j^3 - z_k^3)$, so that $\mathcal{A}(G_{25}) \subset \mathcal{A}(G_{26})$. There are two orbits of hyperplanes in $\mathcal{A}(G_{26})$: one orbit consists of the hyperplanes shared with $\mathcal{A}(G_{25})$, while a second is made up of the 9 hyperplanes defined by $\prod_{1\leqslant j<k\leqslant 3}(z_j^3 - z_k^3)$.

We use Maschke's polynomials [4, p. 326] C_6, C_9, C_{12} and C_{18} to define the invariants for G_{25} and G_{26}. See Table 4 for the definitions of these polynomials.

For convenience, we use the flat invariants $\mathcal{B} = \{f_1, f_2, f_3\}$ of [13, p. 282] as the basic invariants for G.

$$f_1 = C_6, \tag{5.1}$$
$$f_2 = 12C_{12} - 3C_6^2, \tag{5.2}$$
$$f_3 = 96C_{18} + 18C_6^3 - 72C_6C_{12}, \tag{5.3}$$

The exponents of G_{26} are thus {5,11,17}.

Table 4
Definitions of the polynomials C_6, C_9, and C_{12}

$C_6(z_1, z_2, z_3) = z_1^6 + z_2^6 + z_3^6 - 10\left(z_1^3z_2^3 + z_1^3z_3^3 + z_2^3z_3^3\right)$
$C_9(z_1, z_2, z_3) = \left(z_1^3 - z_2^3\right)\left(z_2^3 - z_3^3\right)\left(z_3^3 - z_1^3\right)$
$C_{12}(z_1, z_2, z_3) = \left(z_1^3 + z_2^3 + z_3^3\right)\left[\left(z_1^3 + z_2^3 + z_3^3\right)^3 + 216z_1^3z_2^3z_3^3\right]$
$C_{18}(z_1, z_2, z_3) = \left(z_1^3 + z_2^3 + z_3^3\right)^6 - 540z_1^3z_2^3z_3^3\left(z_1^3 + z_2^3 + z_3^3\right)^3 - 5832z_1^6z_2^6z_3^6$

The close relationship between G_{25} and G_{26} carries over to the Milnor fiber complexes of the two groups. The first invariants for G_{25} and G_{26} are identical, and thus their Milnor fibers are as well. Since $\mathcal{A}(G_{25}) \subset \mathcal{A}(G_{26})$, the Milnor fiber complex for G_{26} can be viewed as a subdivision of the complex for G_{25}.

With our choice of invariants, the discriminant matrix has been calculated (see [13, p. 282]):

$$\mathsf{M}_\Delta(T_1, T_2, T_3; \mathcal{B}) = \begin{pmatrix} 6T_1 & 12T_2 & 18T_3 \\ 12T_2 & 216T_1^3 + 108T_1T_2 + 18T_3 & 864T_1^2T_2 + 72T_2^2 \\ 18T_3 & 864T_1^2T_2 + 72T_2^2 & 7776T_1^5 + 1080T_1T_2^2 \end{pmatrix}.$$

From Corollary 3.1, we have that the image $\Lambda = \pi(\Gamma)$ in the orbit space E_G is a 2-simplex. The topology of Λ is determined by the minors of $\mathsf{M}_\Delta(1, T_2, T_3; \mathcal{B})$. First, the vertices of Λ are the simultaneous solutions of the determinants of the 2×2 minors of $\mathsf{M}_\Delta(1, T_2, T_3; \mathcal{B})$. There are three such vertices. In (T_1, T_2, T_3) coordinates, they are given by

$$\begin{aligned} P &= (1, 9, 42), \\ Q &= (1, -3, 18), \\ S &= (1, 0, -12). \end{aligned}$$

The discriminant polynomial for G_{26} is the determinant of $\mathsf{M}_\Delta(T_1, T_2, T_3; \mathcal{B})$ (up to constant multiple):

$$\begin{aligned} \Delta_G(T_1, T_2, T_3; \mathcal{B}_G) \doteq\ & 10077696T_1^9 - 4199040T_1^5T_2^2 + 5038848T_1^7T_2 \\ & - 46656T_1^3T_2^3 + 839808T_3T_1^6 + 489888T_1^2T_2^2T_3 \\ & - 186624T_1T_2^4 + 31104T_2^3T_3 - 69984T_1^3T_3^2 \\ & - 34992T_1T_2T_3^2 - 5832T_3^3. \end{aligned} \tag{5.4}$$

The zero set of $\Delta_G(1, T_2, T_3)$ gives the union of the 1-cells of Λ, which make up the boundary. Λ is topologically a 2-simplex in the plane $T_1 = 1$, consisting of vertices P, Q, and S, arcs joining these vertices, and the interior points of this triangle (in the real parts of the T_2 and T_3 coordinates). See Fig. 3.

The polynomial $\Delta_G(1, T_2, T_3)$ can be factored to obtain two components of the boundary Λ:

$$16T_2^3 + 36T_2^2 + 432T_2 - 36T_3T_2 - 3T_3^2 + 432 = 0, \quad \text{and} \tag{5.5}$$

$$6T_2 - T_3 - 12 = 0. \tag{5.6}$$

The first component yields the two arcs joining P to Q and Q to S, while the second component gives the line segment joining points P and S in E_G.

To describe the Milnor fiber complex for G_{26}, we construct the complex inductively, beginning with the 0-cells. As noted in Table 1, Γ consists of 342 vertices, 1512 edges and 1296 faces. We find the vertices by intersecting the rank 2 elements of $L(\mathcal{A})$ with F. [13, Table C.7] tells us that there are 57 such rank 2 elements. Thus there are $57 \cdot 6 = 342$ vertices in Γ. The vertices come in three orbits: one orbit consisting of 72 points, one

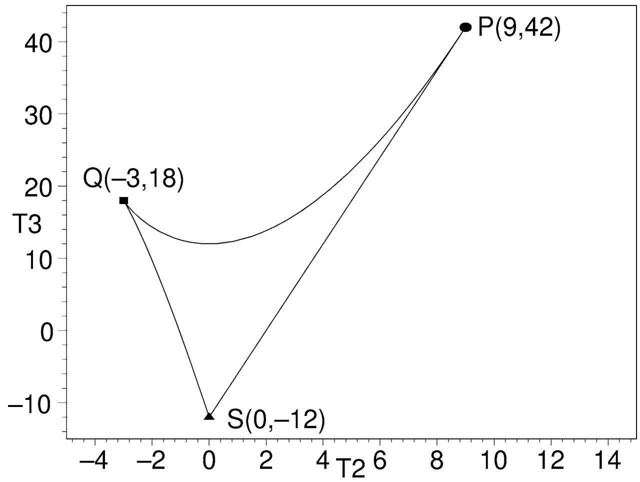

Fig. 3. The image Λ of the Milnor fiber complex for G_{26} in the plane $T_1 = 1$ in the orbit space (projected into coordinates (T_2, T_3)).

consisting of 54 points, and one consisting of 216 points. Let $P_1, \dots, P_{72}$, $Q_1, \dots, Q_{54}$, and $S_1, \dots, S_{216}$ denote the three respective orbits of vertices. Representative elements of the orbits of vertices are given by

$$\begin{aligned} P_1 &= (1, 0, 0), \\ Q_1 &= (c, -c, 0), \\ S_1 &= (2d, -d, -d), \end{aligned}$$

where $c = \frac{1}{\sqrt[6]{12}} \approx 0.6609$ and $d = \frac{1}{\sqrt{6}} \approx 0.4082$. Under the orbit map π, these vertices are projected to their respective vertices in the plane $T_1 = 1$. That is, $\pi(P_j) = (1, 9, 42) = P$, $\pi(Q_j) = (1, -3, 18) = Q$, and $\pi(S_j) = (1, 0, -12) = S$.

We now describe the 1-skeleton of the complex for G_{26}. As noted above, there are two orbits of the hyperplanes of $\mathcal{A}$, so Γ_H does depend on $H \in \mathcal{A}$. Let $H_1 = \ker z_3$ and $H_2 = \ker(z_2 - z_3)$ be representatives of the two orbits. The two orbits of hyperplanes correspond to the two components of the boundary of Λ: $\pi(H_1 \cap F)$ retracts onto the arc defined in Eq. (5.5) while $\pi(H_2 \cap F)$ corresponds to the line segment given in Eq. (5.6).

For $j = 1, 2$, we construct the complex $\Gamma_{H_j} = \Gamma \cap H_j$, which will be a strong deformation retract of $F \cap H_j$. Then the 1-skeleton of Γ is obtained from the union of the orbits of the two subcomplexes. Both subcomplexes have Milnor number $\mu = 25$. Moreover, $\mathcal{A}^{H_1}$ and $\mathcal{A}^{H_2}$ are both arrangements of 8 complex lines. Thus, both Γ_{H_1} and Γ_{H_2} contain $8 \cdot 6 = 48$ vertices. Then, because $\mu = 25$ and $v_0 = 48$ in the subcomplex, we have $v_1 - v_0 + 1 = 25$, or $v_1 = 72$. This gives us the number of edges in each subcomplex.

For $H_1 = \ker z_3$, we have that $F \cap H_1$ is the complex surface in the complex hyperplane $z_3 = 0$ defined by $z_1^6 + z_2^6 - 10 z_1^3 z_2^3 = 1$. The subcomplex Γ_{H_1} is a strong deformation retract of this surface. This subcomplex contains 12 vertices from the P orbit, and 18 vertices from each of the other two orbits. It contains two orbits of the edges of Γ, in particular those edges whose image under the orbit map is either the arc joining P and Q or the arc joining Q and S in Λ. To define these two orbits of edges of the subcomplex, we use two different Γ_{H_1}-semigenerating 2-flats. Let $X_1^{(2)} = H_1 \cap \ker(y_1) \cap \ker(y_2)$, and $X_2^{(2)} = H_1 \cap \ker(x_1 + x_2) \cap \ker(y_1 - y_2)$. Then $X_1^{(2)}$ and $X_2^{(2)}$ are real codimension 4 subspaces of $\mathbb{R}^6$ living in H_1. The Milnor fiber F is defined by two additional real restrictions (The complex restriction $f_1(z_1, z_2, z_3) = 1$ is equivalent to the coupled real restrictions $\operatorname{Re} f_1(x_1+\mathrm{i}y_1, x_2+\mathrm{i}y_2, x_3+\mathrm{i}y_3) = 1$ and $\operatorname{Im} f_1(x_1+\mathrm{i}y_1, x_2+\mathrm{i}y_2, x_3+\mathrm{i}y_3) = 0$). One might expect that $X_1^{(2)} \cap F$ and $X_2^{(2)} \cap F$ are real codimension 6 sets (i.e., sets of points). However, both $X_1^{(2)}$ and $X_2^{(2)}$ are subsets of the space defined by $\operatorname{Im} f_1(x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2, x_3 + \mathrm{i}y_3) = 0$. Thus, $X_1^{(2)} \cap F$ and $X_2^{(2)} \cap F$ are real algebraic curves in $X_1^{(2)}$ and $X_2^{(2)}$, respectively.

Projecting into (x_1, x_2) coordinates, we see that $X_1^{(2)} \cap F$ is the two-branched real curve $x_1^6 + x_2^6 - 10x_1^3x_2^3 = 1$ in $X_1^{(2)}$. The curve contains 6 of the vertices of the complex: four vertices from the P orbit ($P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$, $P_3 = (-1, 0, 0)$, and $P_4 = (0, -1, 0)$), as well as the points $Q_1 = (c, -c, 0)$ and $Q_2 = (-c, c, 0)$, where $c = \frac{1}{\sqrt[6]{12}}$. See Fig. 4.

Topologically, $X_2^{(2)} \cap F$ is the circle $12x_1^6 + 60x_1^2y_1^4 + 8y_1^6 = 1$ in $X_2^{(2)}$ (projecting into the x_1 and y_1 coordinates). This curve contains 12 vertices of the complex, from the Q and S orbits. Two of these vertices are $Q_1 = (c, -c, 0)$ and $Q_2 = (-c, c, 0)$ defined earlier, and there are four other vertices from this orbit (call them Q_3, Q_4, Q_5 and Q_6) distributed evenly around the circle. $X_2^{(2)}$ also contains six vertices from the type S orbit. Let $b = \frac{1}{\sqrt{2}}$. Then $X_2^{(2)}$ contains the vertices $S_2 = (b\mathrm{i}, b\mathrm{i}, 0)$ and $S_5 = (-b\mathrm{i}, -b\mathrm{i}, 0)$ as well as four other vertices from the S orbit (call them S_4, S_5, S_6 and S_7), distributed evenly around the

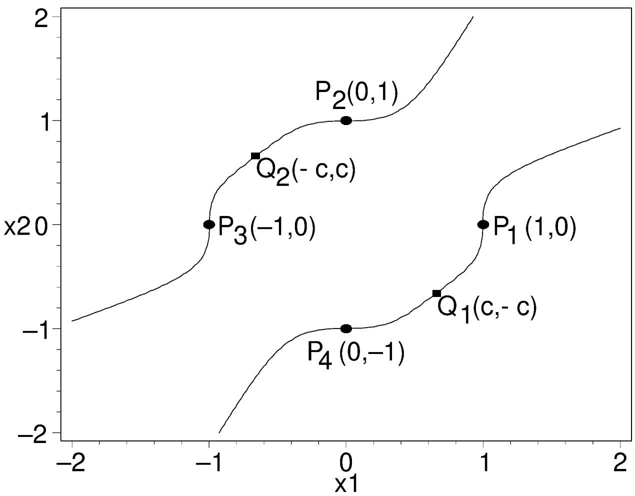

Fig. 4. The projection of the curve $x_1^6 + x_2^6 - 10x_1^3x_2^3 = 1$ in the generating 2-flat $X_1^{(2)} \cap F = \{y_1 = y_2 = z_3 = 0\} \cap F$ onto the (x_1, x_2) plane.

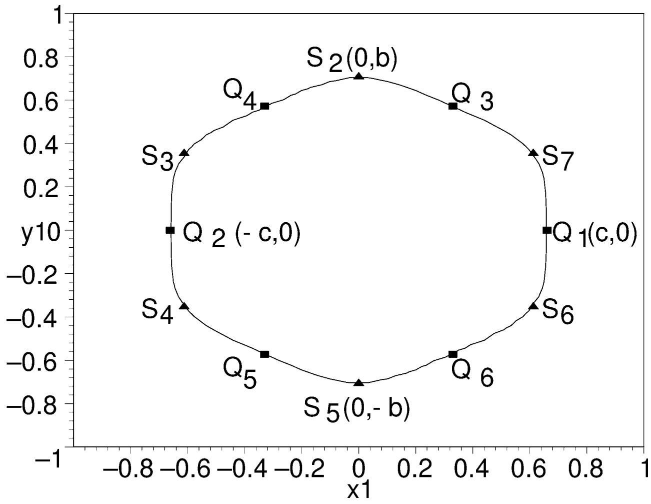

Fig. 5. The projection of the curve $12x_1^6 + 60x_1^2y_1^4 + 8y_1^6 = 1$ in the generating 2-flat $X_2^{(2)} \cap F = \{x_1 + x_2 = y_1 - y_2 = z_3 = 0\} \cap F$ onto the (x_1, y_1) plane.

circle. See Fig. 5 for a picture of $X_2^{(2)} \cap F$, projected into the (x_1, y_1) plane. Note that using (x_1, y_1) coordinates, $Q_1 = (c, 0)$, $Q_2 = (-c, 0)$. $S_2 = (0, b)$, and $S_5 = (0, -b)$.

The arcs $\overline{P_1Q_1}$ in $X_1^{(2)}$ and $\overline{Q_1S_7}$ in $X_2^{(2)}$ are generating arcs for the subcomplex Γ_{H_1}. Let $e_1 = \overline{P_1Q_1}$ and $e_2 = \overline{Q_1S_7}$. To find the remaining edges of this complex, we consider the orbits of $X_1^{(2)}$ and $X_2^{(2)}$ under the action of G. There are nine 2-flats in the orbit of $X_1^{(2)}$ and three 2-flats in the orbit of $X_2^{(2)}$ which lie in H_1. Each 2-flat in the orbit of $X_1^{(2)}$ contains four edges of the complex, so there are 36 edges in the orbit of e_1 in H_1. Each 2-flat in the orbit of $X_2^{(2)}$ contains 12 edges of the complex, so there are 36 edges in the orbit of e_2 in H. This gives us the expected 72 edges of Γ_{H_1}. One can check that $\pi(e_1) = \overline{PQ}$ in Λ, while $\pi(e_2) = \overline{QS}$.

We now consider Γ_{H_2}, where $H_2 = \ker(z2 - z3)$. This subcomplex is a strong deformation retract of $F \cap H_2$, which is the complex surface in H_2 defined by the equation $z_1^6 - 8z_2^6 - 20z_1^3z_2^3 = 1$. Γ_{H_2} contains 24 vertices from the P orbit, and 24 vertices from the S orbit. There is a single orbit of the edges of Γ, specifically, those edges of Γ whose image under the orbit map π is the segment $\overline{PS}$ in Λ. To find these edges, we define a generating 2-flat in H_2. Let $X_3^{(2)} = H_2 \cap \ker(y_1) \cap \ker(y_2)$. Like $X_1^{(2)}$ and $X_2^{(2)}$, the space $X_3^{(2)}$ is contained in the variety defined by $\mathrm{Im}\, f_1 = 0$. Thus $X_3^{(2)} \cap F$ is a real algebraic curve in $X_3^{(2)}$. When projected into (x_1, x_2) coordinates, $X_3^{(2)} \cap F$ is defined by the real equation $x_1^6 - 8x_2^6 - 20x_1^3x_2^3 = 1$. The curve has two branches and contains four of the vertices of the complex: previously defined points $P_1(1, 0, 0)$, $P_2(-1, 0, 0)$, and $S_1(2d, -d, -d)$, together with the vertex $S_8(-2d, d, d)$ (recall $d = \frac{1}{\sqrt{6}}$). See Fig. 6.

The arc $e_3 = \overline{P_1S_1}$ in $X_3^{(2)}$ is a generating arc for Γ_{H_2}. To find the remaining arcs in the subcomplex, we find all 2-flats in the orbit of $X_3^{(2)}$ which lie in H_2. There are 36 such 2-flats, and since each 2-flat contains two edges of the complex, we have the expected 72

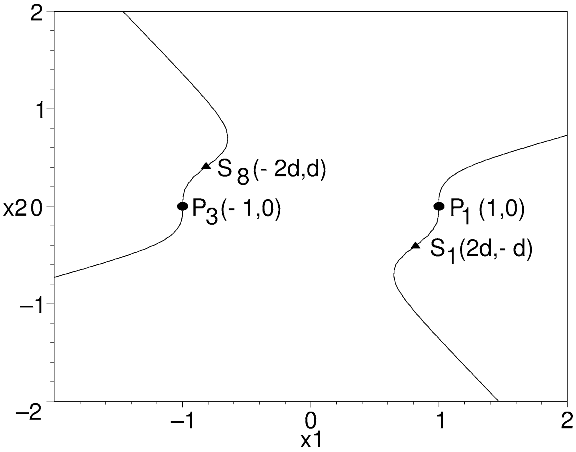

Fig. 6. The curve $x_1^6 - 8x_2^6 - 20x_1^3x_2^3 = 1$ in the generating 2-flat $X_3^{(2)} \cap F = \{y_1 = y_2 = z_2 - z_3 = 0\} \cap F$. The curve has been projected into (x_1, x_2) coordinates.

edges of Γ_{H_2}. One can check that in the orbit space E_G, edge e_3 is mapped to the line segment $\overline{PS}$.

Given our subcomplexes Γ_{H_1} and Γ_{H_2}, we obtain the 1-skeleton of the full complex Γ by considering the action of G on the edges e_1, e_2, and e_3 defined above. There are respectively, 108, 36, and 324 2-flats altogether in the full orbits of the 2-flats $X_1^{(2)}$, $X_2^{(2)}$, and $X_3^{(2)}$. This tells us that there are 432, 432, and 648 edges in the orbits of the edges e_1, e_2 and e_3, respectively. This gives us the 1512 1-cells in Γ and completes the description of the 1-skeleton of Γ.

We complete the description of the complex for G_{26} by finding a 2-simplex which generates the complex. We do this by defining a generating 3-flat satisfying Definition 5.1. Let $X_1^{(3)} = \ker(y_1) \cap \ker(y_2) \cap \ker(y_3)$. $X_1^{(3)}$ is a real codimension 3 subspace of $\mathbb{R}^6$. Just as $X_1^{(2)}$, $X_2^{(2)}$, and $X_3^{(2)}$ are subspaces of the space $\operatorname{Im} f = 0$, so too is $X_1^{(3)}$. Thus, the space $F \cap X_1^{(3)}$ is a real algebraic surface, defined by the equation

$$x_1^6 + x_2^6 + x_3^6 - 10x_1^3x_2^3 - 10x_1^3x_3^3 - 10x_2^3x_3^3 = 1$$

in $X_1^{(3)}$. See Fig. 7.

$X_1^{(3)}$ contains 18 vertices from the complex. Six of these vertices $((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1))$ belong to the P orbit, six $((\pm c, \mp c, 0), (0, \pm c, \mp c), (\pm c, 0, \mp c))$ are from the Q orbit, and six $((\pm 2d, \mp d, \mp d), (\mp d, \pm 2d, \mp d), (\mp d, \mp d, \pm 2d))$ are contained in the S orbit. In particular, note that $P_1, Q_1, S_1 \in F \cap X_1^{(3)}$.

The space $X_1^{(3)}$ contains seven of the generating 2-flats for the 1-skeleton, given by

$$\ker(x_j) \cap Z, \quad \text{for } 1 \leqslant j \leqslant 3, \tag{5.7}$$

$$\ker(x_1 + x_2 + x_3) \cap Z, \quad \text{and} \tag{5.8}$$

$$\ker(x_j - x_k) \cap Z, \quad \text{for } 1 \leqslant j < k \leqslant 3. \tag{5.9}$$

The first three flats belong to the orbit of $X_1^{(2)}$, and thus intersect F in curves similar to Fig. 4. In fact, note that $X_1^{(2)} = \ker(x_3) \cap Z$. The fourth flat defined by $\ker(x_1 + x_2 + x_3) \cap Z$ is in the orbit of $X_2^{(2)}$ and contributes a circle similar to Fig. 5. The remaining three 2-flats lie in the orbit of $X_3^{(2)}$ and intersect the Milnor fiber in curves similar to Fig. 6. In particular, $X_3^{(2)} = \ker(x_2 - x_3) \cap Z$. The curves formed by the intersection of F with these seven 2-flats are shown in Fig. 7 on the surface of $F \cap X_1^{(3)}$. The seven 2-flats thus define the subcomplex of the 1-skeleton of Γ which lies in $X_1^{(3)}$. $X_1^{(3)}$ contains 12 edges each from the orbits of the edges e_1, e_2 and e_3 from this 1-skeleton. Fig. 8 gives two views of these edges (the compact portions of the corresponding algebraic curves) in $X_1^{(3)}$.

The vertices P_1, Q_1, and S_1 span a 2-simplex in F. These three vertices, together with edges e_1, an edge from the orbit of e_2 (specifically, arc $\overline{Q_1 S_1}$ on the meridional circle $\ker(x_1 + x_2 + x_3) \cap Z \cap F$), and edge e_3 form a triangle on the surface of $F \cap X_1^{(3)}$. This

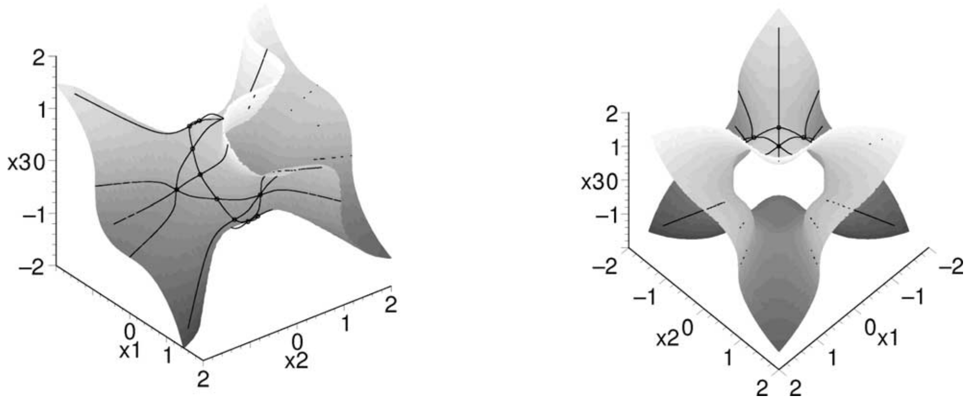

Fig. 7. Two views of the surface $X_1^{(3)} \cap F = \{y_1 = y_2 = y_3 = 0\} \cap F$. In coordinates (x_1, x_2, x_3), the surface has defining equation $x_1^6 + x_2^6 + x_3^6 - 10x_1^3x_2^3 - 10x_1^3x_3^3 - 10x_2^3x_3^3 = 1$. Curves which make up edges of the complex $\Gamma_{G_{26}}$ are also shown on the surface.

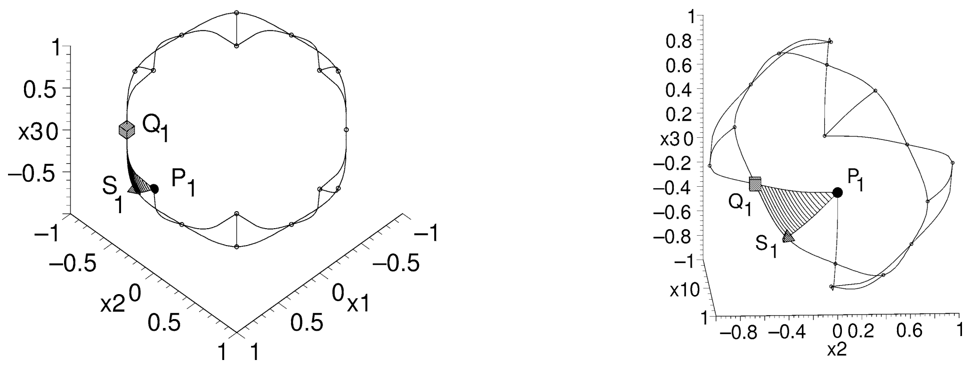

Fig. 8. Two views of the edges of the complex $\Gamma_{G_{26}}$ which lie in the 3-flat $X_1^{(3)} = \{y_1 = y_2 = y_3 = 0\}$.

triangle, together with its interior, forms a fundamental 2-cell $\sigma = \sigma_{G_{26}}$ of $\Gamma_{G_{26}}$. Then $\pi(\sigma) = \Lambda$, i.e., $\pi(\sigma)$ is the triangle $\triangle PQS$ (with its interior) in E_G.

Note that along with σ, there are eleven other 2-cells of Γ lying in $X_1^{(3)}$. When we consider the orbit of $X_1^{(3)}$ under the action of G, we obtain a total of 108 3-flats. Each 2-cell of the complex will be contained in a single 3-flat, so there are 1296 2-cells in Γ, which agrees with Table 1. Since there are 72 vertices in the P orbit, and 1296 2-simplexes, every P vertex lies in eighteen 2-cells. There are 432 edges of each of orbit types e_1 and e_2, so each P vertex lies in six type e_1 edges and six type e_2 edges. Similarly, since there are 648 edges of type e_3, each P vertex lies in nine edges from the orbit of e_3.

In a similar way, each vertex from the Q orbit lies in $\frac{1296}{54} = 24$ of the 2-cells. Each type Q point is an endpoint of eight of the edges from each of the e_1 and e_2 orbits. Each vertex from the S orbit lies in six of the 2-cells of Γ. Each type S point is also an endpoint for two of the edges from the e_1 orbit, two of the edges from the e_2 orbit and three of the edges from the e_3 orbit.

We noted at the start of this description that $\Gamma_{G_{26}}$ is a subdivision of $\Gamma_{G_{25}}$. We can now be more explicit about this relationship. We may choose the same first invariant for G_{25} as for G_{26}, i.e., $f_1 = C_6$. If we use this invariant together with the information given in Table 3, one obtains the complex for G_{25}. A fundamental 2-cell $\sigma_{G_{26}}$ is obtained by bisecting a 2-cell for G_{25} in $X_1^{(3)}$ by the 2-flat $X_3^{(2)}$. Put differently, arc $\overline{P_1 S_1}$ belongs to exactly two triangles in $X_1^{(3)}$. The union of those two triangles is the underlying space of $\sigma_{G_{25}}$. This can be seen in Figs. 8 and 7.

This completes our description of the complex for G_{26}.

5.2. *The Milnor fiber complex for G_{25}*

Here we give a brief sketch of the complex for G_{25}. The group $G = G_{25}$ consists of 648 elements. As a subgroup of $GL(\mathbb{C}^3)$, it can be represented as the group generated by the three reflections r_1, r_2, r_3 found in [15, Eq. 10.3]. The reflection arrangement $\mathcal{A} = \mathcal{A}(G)$ has 12 hyperplanes. A defining polynomial for $\mathcal{A}$ is

$$Q(\mathcal{A}) = z_1 z_2 z_3 \prod_{0 \leqslant j,k \leqslant 2} \left(z_1 + \omega^j z_2 + \omega^k z_3\right).$$

We use the flat invariants $\mathcal{B} = \{f_1, f_2, f_3\}$ of [13, p. 282] as the basic invariants for G:

$$f_1 = C_6, \tag{5.10}$$

$$f_2 = 32\sqrt{3} C_9, \tag{5.11}$$

$$f_3 = 5C_6^2 - 8C_{12}. \tag{5.12}$$

The exponents of G_{25} are $\{5, 8, 11\}$. The Milnor fiber $F = f_1^{-1}(1)$ is homotopy equivalent to a wedge of 2-spheres, where the number of spheres is given by the Milnor number of F, i.e., $m_1^{\ell} = (6-1)^3 = 125$.

With our choice of invariants, the discriminant matrix M_Δ can be found in [13, p. 282] and the zero set of $\det \mathsf{M}_\Delta(1, T_2, T_3; \mathcal{B})$ is the boundary of the image $\Lambda = \pi(\Gamma)$ in E_G.

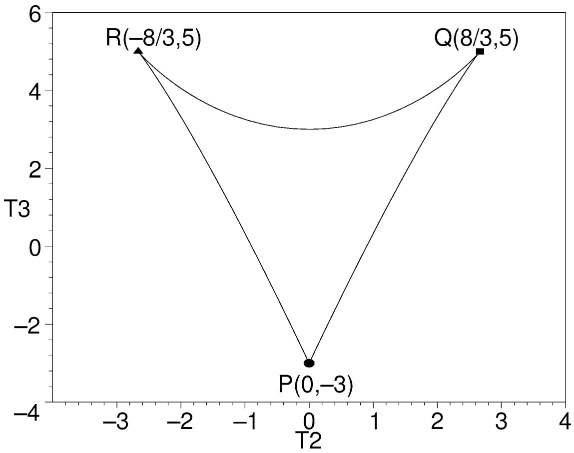

Fig. 9. The image Λ of the Milnor fiber complex for G_{25} in the plane $T_1 = 1$ in the orbit space (projected into coordinates (T_2, T_3)).

Λ is (topologically) a 2-simplex in the plane $T_1 = 1$, with vertices $P = (1, 0, -3)$, $Q = (1, \frac{8}{3}, 5)$, and $R = (1, -\frac{8}{3}, 5)$. See Fig. 9.

The vertices of Γ are obtained by intersecting the rank 2 elements of the lattice with the Milnor fiber. There are 126 such vertices, in three orbits: one orbit consisting of 72 points, and two consisting of 27 points each.

There is a single orbit of hyperplanes in $\mathcal{A}$, so $L(\mathcal{A}^H)$ is independent of H. Let H be the hyperplane defined by $z_3 = 0$. The complex $\Gamma_H = \Gamma \cap H$ is a strong deformation retract of $F^H = F \cap H = \{(z_1, z_2, 0) \mid z_1^6 + z_2^6 - 10z_1^3z_2^3 = 1\}$, and the 1-skeleton of Γ is found by looking at the G orbit of Γ_H.

Let

$$\begin{aligned} X_1^{(2)} &= H \cap \ker(y_1) \cap \ker(y_2), \quad \text{and} \\ X_2^{(2)} &= H \cap \ker(x_1 + x_2) \cap \ker(y_1 - y_2). \end{aligned}$$

Then $X_1^{(2)} \cap F$ is a two-branched algebraic curve containing edges from two orbits in the complex. $X_2^{(2)} \cap F$ is topologically a circle which contains edges of the third orbit type. The compact pieces of these curves (i.e., the portion which lies between vertices) and their G-orbits compose the 1-skeleton of Γ. The curves are respectively the same as those for the complex for G_{26} seen in Figs. 4 and 5, but the vertices are not the same. In particular, the vertices of G_{25} which compose the Q and R orbits in G_{25} make up the Q orbit of vertices in $\Gamma_{G_{26}}$.

Let $X_1^{(3)} = \ker(y_1) \cap \ker(y_2) \cap \ker(y_3)$. Then the space $F \cap X_1^{(3)}$ is a real algebraic surface, defined in $X_1^{(3)}$ by the equation

$$x_1^6 + x_2^6 + x_3^6 - 10x_1^3x_2^3 - 10x_1^3x_3^3 - 10x_2^3x_3^3 = 1.$$

$X_1^{(3)}$ is a Γ-generating 3-flat for the complex, as it is for $\Gamma_{G_{26}}$. The flat contains three 2-flats from the orbit of $X_1^{(2)}$ as well as one of the 2-flats from the orbit of $X_2^{(2)}$. These four 2-flats are precisely the flats

$$\begin{aligned}\ker(x_j) \cap Z, \quad \text{for } 1 \leqslant j \leqslant 3, \text{ and}\\ \ker(x_1+x_2+x_3) \cap Z\end{aligned}$$

given in Eqs. (5.7) and (5.8). $F \cap X_1^{(3)}$ thus contains six 2-cells of Γ, and the G-orbit of one of these 2-cells yields the entire complex.

5.3. The Milnor fiber complex for G_{32}

Here we give a brief sketch of the complex for G_{32}. G_{32} is a group of order 155520 generated by four reflections. We use the generators of Shephard and Todd from Eq. (10.5) of [15], with the roles of coordinates z_1 and z_4, and z_2 and z_3 interchanged. This interchange is necessary in order to utilize the basic invariants (and the discriminant matrix) for G_{32} found in [13, p. 283].

The reflection arrangement $\mathcal{A} = \mathcal{A}(G)$ consists of 40 hyperplanes. A defining polynomial for $\mathcal{A}$ is

$$\begin{aligned}Q(\mathcal{A}) = z_1z_2z_3z_4 \prod_{0\leqslant j,k\leqslant 2}\left(z_1+\omega^j z_2-\omega^k z_3\right)\prod_{0\leqslant j,k\leqslant 2}\left(z_1-\omega^j z_2+\omega^k z_4\right)\\ \times \prod_{0\leqslant j,k\leqslant 2}\left(z_1+\omega^j z_3-\omega^k z_4\right)\prod_{0\leqslant j,k\leqslant 2}\left(z_2+\omega^j z_3+\omega^k z_4\right).\end{aligned}$$

There is a single orbit of the hyperplanes of $\mathcal{A}$.

We use the polynomials f_1, f_2, f_3, and f_4 found in [13, p. 283] as our set $\mathcal{B}$ of basic invariants for the group. See [4, p. 337] for Maschke's definitions of the polynomials F_{12}, F_{18}, F_{24}, and F_{30}:

$$f_1 = F_{12}, \tag{5.13}$$

$$f_2 = \tfrac{4}{3}F_{18}, \tag{5.14}$$

$$f_3 = 21F_{12}^2 - 25F_{24}, \tag{5.15}$$

$$f_4 = \tfrac{8}{5}(11F_{12}F_{18} - 25F_{30}). \tag{5.16}$$

The exponents of G_{32} are $\{11, 17, 23, 29\}$. The Milnor fiber $F = f_1^{-1}(1)$ is homotopy equivalent to a wedge of 3-spheres, where the number of spheres is given by the Milnor number of F, i.e., $m_1^\ell = 14641$.

With our choice of invariants, the discriminant matrix M_Δ can be found in [13, p. 283]. The determinant of M_Δ is the polynomial $\Delta(T_1, T_2, T_3, T_4; \mathcal{B})$, and the image Λ of Γ in the orbit space E_G is the zero set of $\Delta(1, T_2, T_3, T_4; \mathcal{B})$ together with the interior points of this set. Λ is (topologically) a 3-simplex in the plane $T_1 = 1$, with vertices $P = (1, -2\sqrt{6}, 21, -\frac{132}{5}\sqrt{6})$, $Q = (1, 2\sqrt{6}, 21, \frac{132}{5}\sqrt{6})$, $R = (1, -\frac{4}{3}, -4, \frac{112}{5})$, and $S = (1, \frac{4}{3}, -4, -\frac{112}{5})$. See Fig. 10.

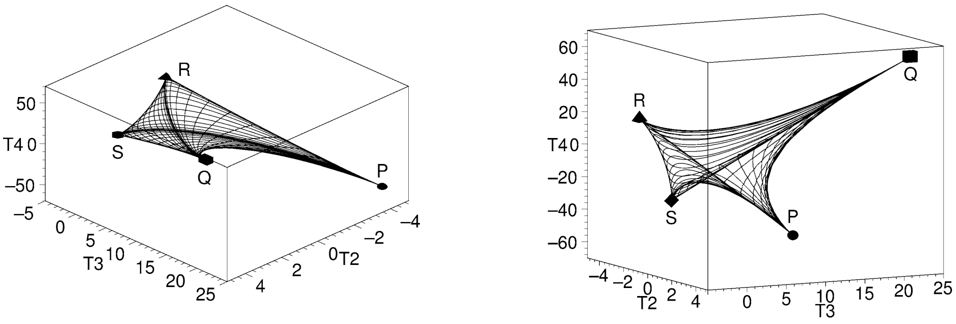

Fig. 10. Two views of the image Λ of the Milnor fiber complex for G_{32} in the plane $T_1 = 1$ in the orbit space (projected into coordinates (T_2, T_3, T_4)).

The vertices of Γ are obtained by intersecting the rank 3 elements of the lattice $L(\mathcal{A})$ with the Milnor fiber. There are 4800 such vertices, in four orbits: two orbits consisting of 240 points each whose images in the orbit space are P and Q, and two consisting of 2160 points each which are projected to R and S in E_G.

There are two orbits of the rank two elements of $L(\mathcal{A})$, represented respectively by the spaces $Z_1 = \ker(z_4) \cap \ker(z_3)$ and $Z_2 = \ker(z_4) \cap \ker(z_2 + z_3)$. When the 1-complexes Γ_{Z_1} and Γ_{Z_2} have been constructed, the 1-skeleton for the full complex Γ may be obtained by considering the orbits of the subcomplexes under the action of G. Since the image $\Lambda = \pi(\Gamma)$ is topologically a tetrahedron, there are six orbits of edges of Γ. Three of those orbits can be found in Γ_{Z_1}, while the remaining three orbits are represented in Γ_{Z_2}.

The restriction $Z_1 = \ker(z_3) \cap \ker(z_4)$ contains 96 of the vertices of the full complex, twelve from each of orbits P and Q, and 36 from each of orbits R and S. To find the edges of Γ_{Z_1}, we define three distinct Γ_{Z_1}-semigenerating 2-flats which intersect F in real algebraic curves. Let

$$\begin{aligned}
X_1^{(2)} &= \ker(y_1) \cap \ker(y_2) \cap Z_1, \\
X_2^{(2)} &= \ker(x_1) \cap \ker(x_2) \cap Z_1, \\
X_3^{(2)} &= \ker\bigl(-\tfrac{1}{2}x_1 + \tfrac{1}{2}\sqrt{3}y_1 - x_2\bigr) \cap \ker\bigl(\tfrac{1}{2}\sqrt{3}x_1 + \tfrac{1}{2}y_1 - y_2\bigr) \cap Z_1.
\end{aligned}$$

Topologically, $X_j^{(2)} \cap F$ is a circle in each case, but each circle contains edges of the complex of a different type. $X_1^{(2)} \cap F$ contains four vertices from each of orbits P and S, and eight edges of the complex connecting the type P and type S vertices. $X_2^{(2)} \cap F$ contains four vertices from each of orbits Q and R, and eight edges of the complex connecting the type Q and type R vertices. $X_3^{(2)} \cap F$ contains six vertices from each of orbits R and S, and twelve edges of the complex connecting the type R and type S vertices. We thus obtain, respectively, three edges e_1, e_2 and e_3 in Γ_{Z_1} which are in distinct G-orbits. To obtain the 1-skeleton of Γ_{Z_1}, we consider all the edges in the orbits of our generating 2-flats which lie in Z_1. In this orbit, there are 72 edges of each type, which yields 216 total edges in Γ_{Z_1}.

We now consider the subcomplex of Γ obtained when we restrict to the subspace $Z_2 = \ker(z_4) \cap \ker(z_2 + z_3)$. Like Γ_{Z_1}, the subcomplex Γ_{Z_2} contains 96 vertices of Γ in four orbits as well as 216 edges of the complex in three orbits. However, the counts of the individual orbit types are different. Z_2 contains 24 vertices of each of the four orbit types. Moreover, the three orbit types of edges can be obtained by considering just a single orbit of 2-flats. Define

$$X_4^{(2)} = \ker(y_1) \cap \ker(y_2) \cap Z_2.$$

The two-branched curve $X_4^{(2)} \cap F$ contains eight of the vertices of the complex, with each branch containing one vertex from each of the four orbits. In this way we obtain the remaining three orbits of edges of Γ: those that connect orbit S points to orbit Q points, those that connect type Q and P points, and those that connect type P and R points (call these edge types e_4, e_5, and e_6, respectively). If we consider the orbits of these edges which lie in Z_2, we obtain 72 edges of each type. To find the 1-skeleton of our full complex Γ, one considers the full G-orbits of our generating 2-flats $X_1^{(2)}, \ldots, X_4^{(2)}$.

There is a single orbit of the hyperplanes of $\mathcal{A}$ and thus, the subcomplex Γ_H is independent of $H \in \mathcal{A}$. We choose $H = \ker z_4$ as a representative. The top-dimensional cells in this subcomplex are (topological) triangles. The vertices and edges of Γ_H have been calculated in the construction of Γ_{Z_1} and Γ_{Z_2} above. Γ_H contains 684 vertices of Γ: 72 vertices from each from the orbits P and Q, and 270 each from the orbits of R and S. The 1-skeleton of Γ_H consists of 4536 edges: 864 edges from each of orbit types e_1, e_2 and e_3, and 648 edges from each of the orbit types e_4, e_5 and e_6.

To completely describe the subcomplex in H, we need to describe the 2-cells. Since Λ is topologically a 3-simplex, there are four orbits of the 2-cells of Γ. Since there is but a single orbit of the hyperplanes of $\mathcal{A}$, all of the orbits of 2-cells of Γ will be represented in Γ_H. To find representatives of each of those orbits, we define Γ_H-semigenerating 3-flats. Let

$$\begin{aligned} X_1^{(3)} &= \ker(y_1) \cap \ker(y_2) \cap \ker(y_3) \cap H, \quad \text{and} \\ X_2^{(3)} &= \ker(x_1) \cap \ker(x_2) \cap \ker(x_3) \cap H. \end{aligned}$$

The 3-flat $X_1^{(3)}$ contains 26 vertices of the complex, with representatives from each of the four orbits of points: six vertices from each of orbit types P and R, two from orbit type Q, and twelve from the S orbit. The 3-flat $X_1^{(3)}$ contains representatives for all of the orbits of the edges of the complex *except* for e_2 (edges joining Q and R points). In particular, then, $X_1^{(3)}$ contains 2-simplexes for Γ_H of two different orbit types: simplexes of type $\triangle PQS$ which are spanned by vertices in the P, Q, and S orbits and edges from the e_1, e_4 and e_5 orbits; and simplexes of type $\triangle PRS$ which are spanned by vertices in the P, R, and S orbits and edges from the e_1, e_3 and e_4 orbits.

The 3-flat $X_2^{(3)}$ also contains 26 vertices of the complex: six vertices from each of orbit types Q and S, two from orbit type P, and twelve from the R orbit. $X_1^{(3)}$ contains representatives for all of the orbits of the edges of the complex *except* for e_1 (edges joining P and S points). In particular, then, $X_2^{(3)}$ contains 2-simplexes for Γ_H of two different orbit types: simplexes of type $\triangle PQR$ which are spanned by vertices in the P, Q, and R orbits and edges from the e_2, e_5 and e_6 orbits; and simplexes of type $\triangle QRS$ which are spanned by vertices in the Q, R, and S orbits and edges from the e_2, e_3 and e_4 orbits.

Altogether, the 3-flat $X_1^{(3)}$ contains twelve of the triangles in the $\triangle PQS$ orbit, and 12 of the triangles in the $\triangle PRS$ orbit. Similarly, the 3-flat $X_2^{(3)}$ contains twelve of the triangles in the $\triangle PQR$ orbit, and 12 of the triangles in the $\triangle QRS$ orbit. To finish the construction of Γ_H, we find the triangles in the orbits of these generating triangles which also lie in H. There are 108 3-flats in each of the orbits of $X_1^{(3)}$ and $X_2^{(3)}$. Since each of these 3-flats contains twelve triangles of each of two orbit types, we have that there are 1296 triangles of each orbit type in Γ_H. Thus there are 5184 2-cells in the subcomplex Γ_H. To find the 2-skeleton of the complex Γ, we need to consider the full orbits of our triangles under the action of G. There are 4320 3-flats in each of the orbits of $X_1^{(3)}$ and $X_2^{(3)}$. Thus, there are 51840 2-cells of each of the four orbit types of 2-cells in the complex. This gives a total of 207360 2-cells in Γ, agreeing with our cell count in Table 1. This completes our description of Γ_H as well as the 2-skeleton of Γ.

To complete our description of the full complex for G_{32}, we need to find a 3-cell of the complex. We do this by finding a Γ-generating 4-flat. Let

$$X_1^{(4)} = \bigcap_{j=1}^{4} \ker(y_j).$$

Then $X_1^{(4)}$ is contained in the subspace of $\mathbb{C}^4 \approx \mathbb{R}^8$ defined by

$$\operatorname{Im} f_1(z_1, z_2, z_3, z_4) = 1.$$

Thus $X_1^{(4)} \cap F$ is a real 3-dimensional hypersurface in $X_1^{(4)}$ defined by

$$f_1(x_1, x_2, x_3, x_4) = 1.$$

This 4-flat contains 64 vertices of the complex, including 8 from each of the orbit types P and Q, and 24 from each of the orbit types R and S. The flat contains edges of all six orbit types as well as 2-simplexes from each of the four orbits: $\triangle PQR$, $\triangle PQS$, $\triangle PRS$, and $\triangle QRS$. Finally, we note from Table 1 that Γ contains 155520 3-cells, each of which is topologically a 3-simplex σ for which $\pi(\sigma) = \Lambda$. The hypersurface $X_1^{(4)} \cap F$ contains 36 3-cells of this type. There are 4320 4-flats in the orbit of $X_1^{(4)}$, and thus the 4-flat generates the complex Γ.

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